# Exploring Fourier Transform Techniques with Mathcad
# Document 3: Fourier Transform of a Pulse

by

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**Note**: the Automatic Calculation option in the Math menu should NOT be checked.

## Objective

After completing this exercise, the student should be able to give quantitative examples of the inverse relationship between spectral width and sampling time.

## Introduction

The purpose of this document is to demonstrate the fact that it is impossible for a short sample of a wave to be monochromatic.  For example, while a continuous-wave laser can have an extremely narrow frequency distribution, a short pulse of laser radiation must contain a range of frequencies.

## **Part 1**: Sine wave

In this program, we compare the FT (frequency spectrum) of a sine wave with that of a short section of the same sine wave. For convenience in implementing the pulse, the total sampling time is fixed at 0.5s; therefore the number of samples causes the sampling interval to change.

$\nu := 100$ — fundamental frequency (units $s^{-1}$)

$A \equiv .5$ — total sampling time (units: s)

$\frac{A}{m} = 4.883 \cdot 10^{-4}$ — sampling interval

$i := 0 .. m - 1$ — range variable for samples

$t_i := \frac{i}{m} \cdot A$ — time of each sample

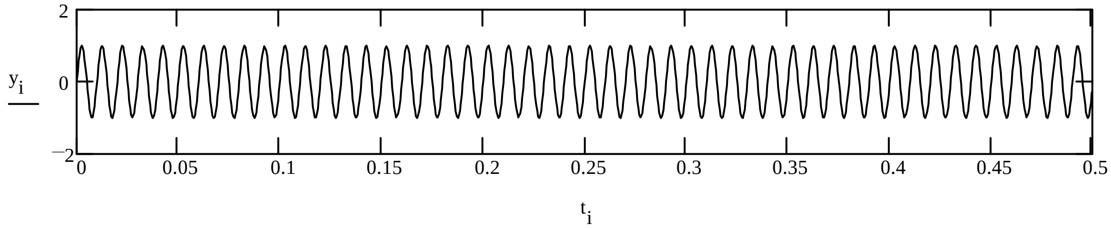
SINE WAVE

$y_i := \sin(2 \cdot \pi \cdot \nu \cdot t_i)$

**EXERCISE**

3.1 Predict the appearances of Graphs 1 and 2 before pressing F9.

**Graph 1**: The wave.

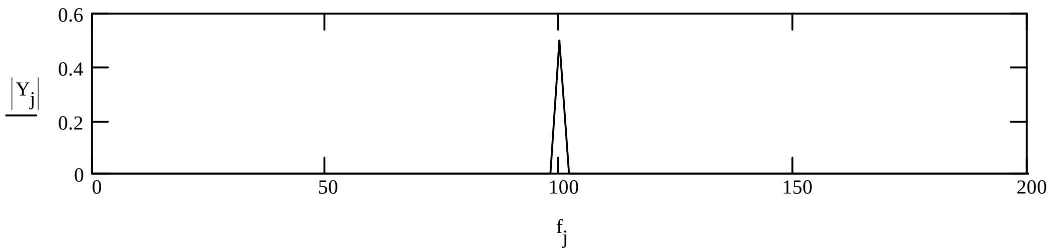



$Y := FFT(y)$ Fourier transform of y

$j := 0 .. \frac{m}{2}$ range variable for FT

$f_j := \frac{j}{t_{m-1}}$ frequency

**Graph 2**: Frequency spectrum of the wave.



**Part 2**: Pulse

Now, we take the same sine wave as above but look only at a short section of it. From one point of view, the shorter sampling time does not provide enough information for the frequency to be found as accurately, so the frequency spectrum will be broadened. Another point of view is that many different frequency components must be present in order for the sum to equal zero outside the width of the pulse. This shows up clearly in the Fourier transform of the pulse.

The notch function used to mask the sine function equals zero except during a short period when it equals one. Other functions could be used; one important alternative is explored in the exercises below.

$k \equiv 10$

$m \equiv 2^k$ m = number of data points in sample

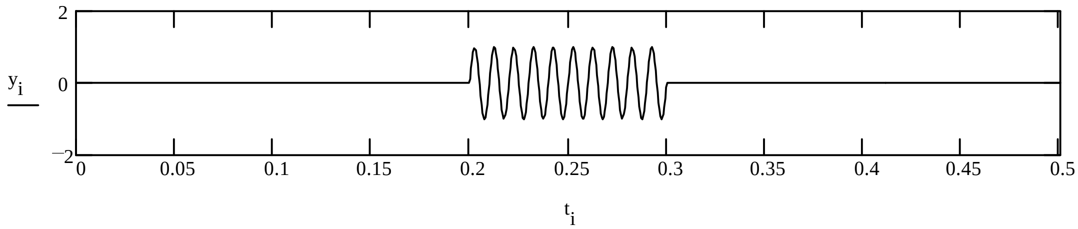
$\delta \equiv .1$ pulse half-duration, expressed as a fraction of total sample time

$\text{notch}(i) := (i > m \cdot (.5 - \delta)) \cdot (i < m \cdot (.5 + \delta))$ notch function

**Mathcad note**
The logical function (a>b)·(a<c) returns a value of 1 if b<a<c and zero otherwise.

$y_i := \sin(2 \cdot \pi \cdot \nu \cdot t_i) \cdot \text{notch}(i)$

**Graph 3**: Pulse

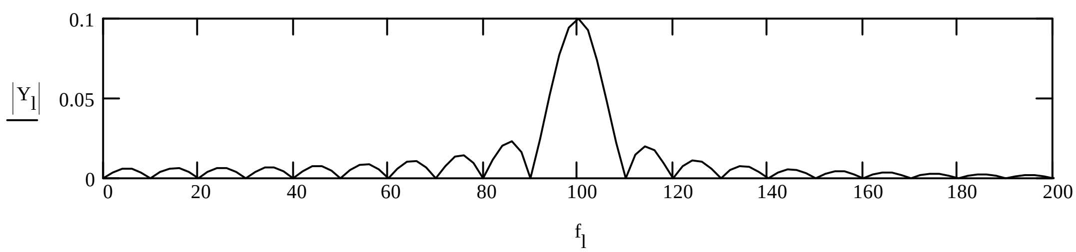



Press F9 to calculate the graph. Note that the effect of the notch function is to permit nonzero values only in the range A/2 ± δ

## **Part 3**: FT of the pulse

$Y := \text{FFT}(y)$ $\quad l := 0 .. \frac{m}{2}$

Graph 4: frequency spectrum of pulse




**EXERCISES**

3.2. Change pulse duration by changing δ and describe the effect on the frequency spectrum.

3.3. Allow Graph 4 to autoscale its x-axis. Explain the maximum frequency appearing in the graph.

**Mathcad note**
To allow a graph to autoscale the x-axis, click on the graph, then on the scale limit below and to the left of the x-axis.
Press F3, then F9.

3.4 Why does the graph of the FT of the sine wave look like a triangle in Graph 2? (Hint: change the formatting of the graph to plot symbols instead of lines.) What is the resolution of this spectrum?

**Mathcad note**
Double-click the graph. In the Traces menu, give trace 1 a symbol, change its line to "none" and click OK.

3.5  Explain the statement, "ultrashort laser pulses cannot be monochromatic."
3.6  In the equation for $y_i$ immediately before Graph 3, replace the notch function with the Gaussian function (shown below).  Describe the resulting FT.
3.7  Obtain an approximate formula relating the Gaussian pulse duration to its width and test it by changing b.

Mathcad note
To replace the notch function with the Gaussian, click on "notch(i)" in the definition of y above Graph 3.  Advance the cursor until it reaches the end of "notch(i)", which should then be underlined in blue.  Press F3 to make "notch(i)" disappear. Type "Gaussian(i)" in its place.

Gaussian function

$$\mathrm{Gaussian}(i) \equiv \exp\left[-\left[\left(\frac{m}{2}-i\right)\cdot b\right]^2\right]$$

Full width at half maximum of Gaussian function/s

$$b \equiv .01 \qquad \frac{2\cdot\sqrt{\ln(2)}\cdot A}{b\cdot m} = 0.081$$

3.8   Calculate the range of frequencies which must be present in a 1 microsecond RF pulse centered at 100 MHz.
3.9   Calculate the range of wavelengths which must be present in a 100 femtosecond laser pulse centered at 600 nm.