Qualitative features of wavefunctions

The Schrodinger Equation (in one dimension, for a particle of mass μ) is

$$\frac{-hbar^2}{2\cdot\mu}\cdot\frac{d^2}{dx^2}\Psi(x) + V(x)\cdot\Psi(x) = E\cdot\Psi(x) \qquad (1)$$

Rearranging gives

$$\frac{-hbar^2}{2\cdot\mu}\cdot\frac{d^2}{dx^2}\Psi(x) = (E - V(x))\cdot\Psi(x) \qquad (2)$$

To look at the qualitative behavior of the solutions, consider the case where V(x) = V, a constant, so we have the differential equation

$$\frac{-hbar^2}{2\cdot\mu}\cdot\frac{d^2}{dx^2}\Psi(x) = (E - V)\cdot\Psi(x) \qquad (3)$$

or $$\frac{d^2}{dx^2}\Psi(x) = \frac{-2\cdot\mu}{hbar^2}(E - V)\cdot\Psi(x)$$

This has the form

$$\frac{d^2}{dx^2}\Psi(x) = a^2\cdot\Psi(x)$$ where

$$a = \sqrt{\frac{-2\cdot\mu}{hbar^2}\cdot(E - V)}$$

The solutions to this equation are

$$\Psi(x) = c1\cdot e^{-a\cdot x} + c2\cdot e^{a\cdot x}$$

If E < V (the wavefunction is tunneling), then a is real, so the solution is a decaying exponential. (the solution that corresponds to an increasing exponential can't be normalized and is unphysical, so its coefficient c is zero).

If a is small, then the exponential decays slowly, and there is more tunneling.
So, tunneling is enhanced if E is not much less than V and for lighter particles.

If E > V, then a is imaginary, so the solutions are

$$\Psi(x) = c1\cdot\cos(|a|\ x) + c2\cdot\sin(|a|\cdot x)$$

where c1 and c2 are constants, and can be complex.

That is, the wavefunction oscillates with a frequency proportional to the square root of (E-V)